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Quantum State Transfer Via a Two-Qubit Heisenberg XYZ Spin Model

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Abstract We report the transfer fidelities of two-qubit pure state via a two-qubit Heisenberg XYZ spin model under a nonuniform magnetic field. It shown us that not only the average fidelity can be enhanced evidently but also the ideal fidelity region can be broadened by introducing the interaction of z component of two neighboring spins J_z. Decreasing the anisotropy parameter γ also can improve F_a and result in ideal fidelity. Our study on the average fidelity of this quantum channel system also shows that for any finite value of temperature T we can obtain the ideal average fidelity by improving the inhomogeneous magnetic field b. F_a is infinitely close to the maximum value of classical communication 2/3 with increasing b. While increasing the uniform field B can not do this, F_a are always inferior to 2/3 for enough larger B.

Keywords Teleportation · Heisenberg chain · Nonuniform magnetic field

1 Introduction

As a valuable resource in quantum information and quantum computation [1, 2], quantum entanglement has attracted numerous attention over past decade years. An entangled composite system gives rise to nonlocal correlation between its subsystems that does not exist classically. This nonlocal property enables the uses of local quantum operations and classical communication to transmit an unknown state via a shared pair of entangled particles with fidelity better than any classical communication protocol, which has been extensively studied both experimentally and theoretically in the past few years [3–5]. The teleportation through some solid systems such as quantum chains is an important emerging field [6–8]. A quantum chain also referred as spin chain is a one-dimensional array of qubits which are coupled permanently by mutual interaction, which can be used to teleport a quantum

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state. Recently, thermally entangled state of a two-qubit Heisenberg chain has been considered as a quantum channel in many papers [9–11]. Cai et al. have investigated the fidelity of quantum communication through spin channels under the influence of decoherence [12]. Reference [10] studies the effects of spin orbit coupling on quantum state transfer through a two-qubit Heisenberg XXX spin chain. In Ref. [13] the author studied the influence of magnetic field and anisotropy on quantum teleportation via a Heisenberg XY chain, but there the magnetic field is uniformly distributed. Liu et al investigated the quantum state transfer fidelity via a two-qubit Heisenberg XXZ chain under an inhomogeneous magnetic field [14]. In papers [15–17] the authors found that there are many nontrivial features about the thermal entanglement of a Heisenberg chain under a nonuniform magnetic field. However, the teleportation via a two-qubit Heisenberg XYZ chain under a nonuniform magnetic field has not been discussed.

Therefore, in this paper we investigate the quantum state transfer via a two-qubit Heisenberg XYZ spin channel under an inhomogeneous magnetic field. By introducing the anisotropy parameter and the interaction of z component of two neighboring spins J_z, we study the influence of these parameters on the transfer fidelity through this transfer channel system. This paper is organized as follows. In Sect. 2, we give the Hamiltonian of our channel system (briefly review the standard teleportation protocol), deduce the output state and investigate the properties of the average fidelity. We conclude our results in Sect. 3.

2 Theoretical Treatment and Results

The channel Hamiltonian of a two-qubit anisotropic Heisenberg XYZ chain under an external uniform magnetic field B and an inhomogeneous magnetic field b along the Z-axis is

$$H = J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + J\gamma(\sigma_1^+\sigma_2^+ + +\sigma_1^-\sigma_2^-) + \frac{J_z}{2}\sigma_1^z\sigma_2^z + \frac{(B+b)}{2}\sigma_1^z + \frac{(B-b)}{2}\sigma_2^z, \tag{1}$$

where $J = \frac{(J_x+J_y)}{2}$, $\gamma = \frac{J_x-J_y}{J_x+J_y}$ and $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$. Among these parameters $(\sigma_i^x,\sigma_i^y,\sigma_i^z)$ are the vectors of Pauli matrices, and J is the real coupling coefficient. The parameter $J > 0$ means that the chain is antiferromagnetic, and ferromagnetic for $J < 0$. The magnetic field on the two-qubit are $B + b$ and $B - b$, respectively, the value of b controls the degree of inhomogeneity. γ $(0 < \gamma < 1)$ denotes the anisotropy in the XY plane. Based on the standard basis states $\{|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\}$, the Hamiltonian can be written as the following matrix form

$$H = \begin{pmatrix} \frac{J_z}{2}+B & 0 & 0 & J\gamma \\ 0 & -\frac{J_z}{2}+b & J & 0 \\ 0 & J & -\frac{J_z}{2}-b & 0 \\ J\gamma & 0 & 0 & \frac{J_z}{2}-B \end{pmatrix}. \tag{2}$$

Without loss of generality, after some straightforward calculations, the eigenvectors of this channel Hamiltonian are easily obtained as following forms

$$|\psi\rangle_1 = N^+\left[\frac{(b+\xi)}{J}|01\rangle + |10\rangle\right],$$

$$|\psi\rangle_2 = N^-\left[\frac{(b-\xi)}{J}|01\rangle + |10\rangle\right], \tag{3}$$

$$|\psi\rangle_3 = M^+\left[\frac{(B+\eta)}{J\gamma}|00\rangle + |11\rangle\right],$$

$$|\psi\rangle_3 = M^-\left[\frac{(B-\eta)}{J\gamma}|00\rangle + |11\rangle\right].$$

with the corresponding energies of this channel system are

$$E_1 = -\frac{J_z}{2} + \xi, \qquad E_2 = -\frac{J_z}{2} - \xi, \qquad E_3 = \frac{J_z}{2} + \eta, \qquad E_4 = \frac{J_z}{2} - \eta. \tag{4}$$

where $\eta = \sqrt{B^2 + J^2\gamma^2}$ and $\xi = \sqrt{b^2 + J^2}$, the normalization constants are $N^\pm = 1/\sqrt{1 + (b \pm \xi)^2/J^2}$ and $M^\pm = 1/\sqrt{1 + (B \pm \eta)^2/J^2\gamma^2}$.

For a spin system in equilibrium at temperature T, the density matrix is $\rho = (1/Z)\times \exp(-H/k_BT)$, where H is the Hamiltonian of this system, Z is the partition function and k_B is the Boltzmann constant. Usually we write $k_B = 1$. By knowing the exact values of eigenvalues and the eigenvectors, then we can calculate the density matrix of this channel system in equilibrium (temperature T). It can be explained as

$$\rho(T) = \begin{pmatrix} \mu_1 & 0 & 0 & v \\ 0 & \omega_1 & y & 0 \\ 0 & y & \omega_2 & 0 \\ v & 0 & 0 & \mu_2 \end{pmatrix}.$$

The exact values of these nonzero matrix elements can be obtained by knowing the spectrum of H. We obtain

$$\begin{aligned}
\mu_1 &= \frac{1}{Z}e^{-\frac{J_z}{2}\beta}\left[\cosh(\eta\beta) - \frac{B}{\eta}\sinh(\eta\beta)\right],\\
\mu_2 &= \frac{1}{Z}e^{-\frac{J_z}{2}\beta}\left[\cosh(\eta\beta) + \frac{B}{\eta}\sinh(\eta\beta)\right],\\
y &= -\frac{J}{Z\xi}e^{\frac{J_z}{2}\beta}\sinh(\xi\beta),\\
\omega_1 &= \frac{1}{Z}e^{\frac{J_z}{2}\beta}\left[\cosh(\xi\beta) - \frac{b}{\xi}\sinh(\xi\beta)\right],\\
\omega_2 &= \frac{1}{Z}e^{\frac{J_z}{2}\beta}\left[\cosh(\xi\beta) + \frac{b}{\xi}\sinh(\xi\beta)\right],\\
v &= -\frac{J\gamma}{Z\eta}e^{-\frac{J_z}{2}\beta}\sinh(\eta\beta).
\end{aligned} \tag{5}$$

where the partition function Z is given by

$$Z = 2[e^{-\frac{J_z}{2}\beta}\cosh(\eta\beta) + e^{\frac{J_z}{2}\beta}\cosh(\xi\beta)]. \tag{6}$$

Based on knowing the density matrix of the channel system, we consider the quantum state transfer through it. The standard teleportation through an entangled mixed state resource can be regarded as a general depolarizing channel [18, 19]. Similar to the standard teleportation protocol, the state transfer for the mixed channel of an input entangled state is destroyed and its replica state appears at the remote place after applying local measurement in the form of linear operators. To see more clearly the quantum state teleportation of two qubits, in this paper the initial input state is assumed to be an entangled two-body pure spin-1/2 state which has the following form

$$|\varphi_{\rm in}\rangle = \cos(\theta/2)|10\rangle + e^{i\phi}\sin(\theta/2)|01\rangle \tag{7}$$

where ($0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$). Here the different values of θ describe all states with different amplitudes, and ϕ are the phase of these states. The output state is given by [10, 20]

$$\rho_{\rm out} = \sum_{ij=0}^{3} P_{ij}(\sigma_i \otimes \sigma_j)\rho_{\rm in}(\sigma_i \otimes \sigma_j), \tag{8}$$

where σ_0 is the identity matrix and σ_i ($i = 1, 2, 3$) is the three components of the Pauli matrix. P_{ij} =Tr$[E^i\rho(T)]$tr$[E^j\rho(T)]$, $\sum_{ij} P_{ij} = 1$ and $\rho_{\rm in}$ is the density matrix of input state. Here $E^0 = |\Psi^-\rangle\langle\Psi^-|$, $E^1 = |\Phi^-\rangle\langle\Phi^-|$, $E^2 = |\Phi^+\rangle\langle\Phi^+|$, $E^3 = |\Psi^+\rangle\langle\Psi^+|$, where $|\Psi^\pm\rangle = (1/\sqrt{2})(|01\rangle \pm |10\rangle)$ and $|\Phi^\pm\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$ respectively. After some straightforward algebra, we obtain the output density matrix has the form of

$$\rho_{\rm out} = \begin{pmatrix} \alpha & 0 & 0 & \delta \\ 0 & y_1 & y_2 & 0 \\ 0 & y_3 & y_4 & 0 \\ \delta & 0 & 0 & \alpha \end{pmatrix}.$$

These nonzero matrix elements are expressed as following

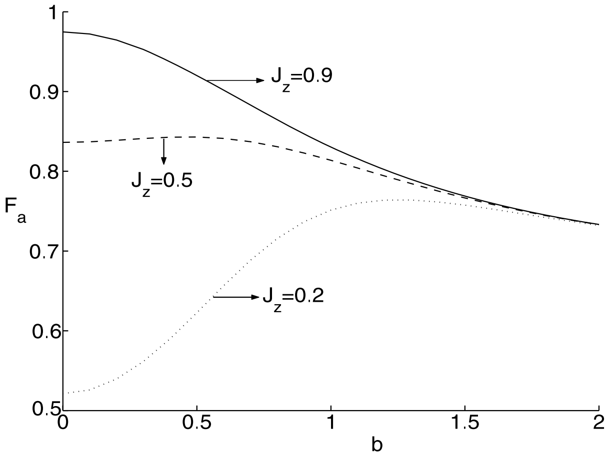
$$\begin{aligned}
\alpha &= (\mu_1 + \mu_2)(\omega_1 + \omega_2), \\
\delta &= 2(\mu_1 + \mu_2)(\omega_1 + \omega_2)\cos(\phi)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right), \\
y_1 &= (\omega_1 + \omega_2)^2\sin^2\left(\frac{\theta}{2}\right) + (\mu_1 + \mu_2)^2\cos^2\left(\frac{\theta}{2}\right), \\
y_2 &= 4y^2e^{i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + 4v^2e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right), \\
y_3 &= 4y^2e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + 4v^2e^{i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right), \\
y_4 &= (\omega_1 + \omega_2)^2\cos^2\left(\frac{\theta}{2}\right) + (\mu_1 + \mu_2)^2\sin^2\left(\frac{\theta}{2}\right).
\end{aligned} \tag{9}$$

By knowing the input states and the output states, to characterize the quality of the teleported state $\rho_{\rm out}$, it is often quite useful to look at the fidelity between $\rho_{\rm in}$ and $\rho_{\rm out}$, which is a measure of the quality of such a spin quantum channel defined by [21]

$$F(\rho_{\rm in}, \rho_{\rm out}) = \{{\rm tr}[\sqrt{(\rho_{\rm in}^{1/2})\rho_{\rm out}(\rho_{\rm in}^{1/2})}]\}^2. \tag{10}$$

The value of fidelity $F(\rho_{\rm in}, \rho_{\rm out})$ ranges from zero to one. When $F(\rho_{\rm in}, \rho_{\rm out}) = 0$, it is means that the information is completely distorted in the transmission process, the quantum state transfer is failed. While for $F(\rho_{\rm in}, \rho_{\rm out}) = 1$, that is to say the final state is identical to the

Fig. 1 With the coupling constant $J = 1$, for a finite value of uniform magnetic field $B = 0.5$. The average fidelity is plotted as a function of nonuniform magnetic field b with different values of J_z at a finite temperature $T = 0.2$. Where we set $\gamma = 0.9$



initial state, thus denoting the ideal communication transmission process. In common situation, $0 < F < 1$, information is distorted in some extent after being transmitted. For quantum communication, F can be larger than 2/3, which is the maximum of classical communication, so that in order to transmit $|\varphi_{in}\rangle$ with better fidelity than any classical communication protocol, we require the value of F to be strictly greater than 2/3. After a straightforward calculation, the fidelity is given by

$$\begin{aligned} F(\rho_{in}, \rho_{out}) = {} & (\omega_1 + \omega_2)^2\left(\sin^4\left(\frac{\theta}{2}\right) + \cos^4\left(\frac{\theta}{2}\right)\right) \\ & + 8\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)[y^2 + v^2\cos(2\phi)] \\ & + 2(\mu_1 + \mu_2)^2\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right). \end{aligned} \tag{11}$$

Since the transported state is a pure state through this spin entangled channel, the efficiency of quantum communication is characterized by the average fidelity, which can be obtained by averaging $F(\rho_{in}, \rho_{out})$ over all possible input states in the Bloch sphere. For our model the average fidelity F_a can be formulated as

$$F_a = \frac{1}{4\pi}\int F d\Omega = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} F(\rho_{in}, \rho_{out})\sin(\theta)d\theta}{4\pi} \tag{12}$$

through some straightforward algebra, one can obtain the average fidelity has the following form

$$F_a = \frac{4}{3Z^2}\left[2e^{J_z\beta}\cosh^2(\xi\beta) + e^{-J_z\beta}\cosh^2(\eta\beta) + \frac{J^2}{\xi^2}e^{J_z\beta}\sinh^2(\xi\beta)\right]. \tag{13}$$

With knowing the transfer fidelities of quantum states via this two-qubit Heisenberg XYZ spin channel model. Next, we will concentrate on our attention to investigate the effects of these parameters on the average fidelity, such as the effects of the inhomogeneous magnetic field b, the anisotropy parameter γ, and the interaction of the z component J_z about this two-qubit transfer state channel system.

In Fig. 1, we give the plot of the average fidelity F_a as a function of the inhomogeneous magnetic field b with different values of J_z at a finite temperature $T = 0.2$, where we set $J = 1$, $\gamma = 0.9$, and the external uniform magnetic field $B = 0.5$. From the figure, we note that the inhomogeneous magnetic field b not only can improve the average fidelity but also can decrease it, which is decided by the value of the interaction J_z. It can be seen clearly that the initial value of F_a is smaller than the maximum of classical communication 2/3 for a smaller value of J_z (such as $J_z = 0.2$, the initial F_a is about 0.52), however, with increasing b, the value of F_a is improved and can be larger than 2/3. So, for a little value of J_z, increasing b can improve the transfer fidelity F_a and finally make the F_a is above 2/3. While for a larger value of J_z (such as $J_z = 0.5$ or 0.9), the value of F_a is strictly greater than 2/3 no matter whatever the value of b is changed. We also observe that increasing the interaction J_z can enhance the transfer average fidelity evidently for finite value of b. The transfer state fidelity of this two-qubit quantum channel are controllable by adjusting J_z and external field. So one can obtain a better fidelity through our channel system by adjusting J_z and the external inhomogeneous magnetic field b.

To clearly see the effects of the anisotropy parameter γ. In Fig. 2, we plot the transfer state average fidelity versus b with different values of γ at different temperatures T. For all plots we set $J = 1$, the external uniform field $B = 0.5$ and $J_z = 0.9$. From these two figures we can see clearly that with the value of γ is decreased the average fidelity can be increased for any finite temperature. For the top panel, the transfer state average fidelity are always superior to classical information channel with increasing b. We also observe that the larger the value of γ, the region where the average fidelity is superior to classical information channel 2/3 is narrower. When the value of b is farther increased, these two lines with different γ are superposable, and the average fidelity is infinitely close to the classical fidelity 2/3 (it is about 0.667), which also can be seen in the bottom panel. So, one can deduce that no matter for whatever finite value of temperature T, we can obtain a better transfer state fidelity through increasing the inhomogeneous magnetic field b. Similar to the case of the top panel, for the bottom panel, decreasing the anisotropy parameter γ also can increasing the average fidelity F_a. However, the value of F_a are always inferior to the maximum of classical communication 2/3. Only for the case of enough large b, F_a is infinitely approaches to the maximum value of classical fidelity 2/3. Comparing these two figures we know that the transfer fidelity is mainly decided by the temperature, F_a is decreased evidently with increasing the temperature T, and increasing b can make F_a infinitely approaches to the maximum value of classical communication 2/3 for any temperatures T and any values of γ.

The average fidelity F_a as a function of uniform magnetic field B with different values of interaction J_z at a finite temperature $T = 0.2$ is shown in Fig. 3, where we set $J = 1$, $\gamma = 0.9$ and the nonuniform magnetic field $b = 0.5$. we tend to farther study the effect of the J_z on the transfer state fidelity via this quantum channel. From the figure, we can see clearly that with the uniform magnetic field B increases the average fidelity is initially decreased to a minimum value and then have a increasing tendency. When B is farther increased, these three lines are superposed and tend to be a constant value. But this constant value is always inferior to 2/3, which is different the effect of b, for enough larger value of b, F_a are always tend to be larger than 2/3. Then we examine the effect of the interaction J_z. The figure shown us that improving the value of interaction J_z can enhance the transfer state average fidelity. The stronger of the interaction J_z, the larger of the maximum value of F_a (for $J_z = 0.2$ the maximum value of F_a is 0.65, but for $J_z = 0.9$ it is about 0.92). Obviously, a larger value of J_z not only can make the value of F_a superior to the maximum value of classical communication 2/3 but also can broaden the region where F_a is larger than 2/3.

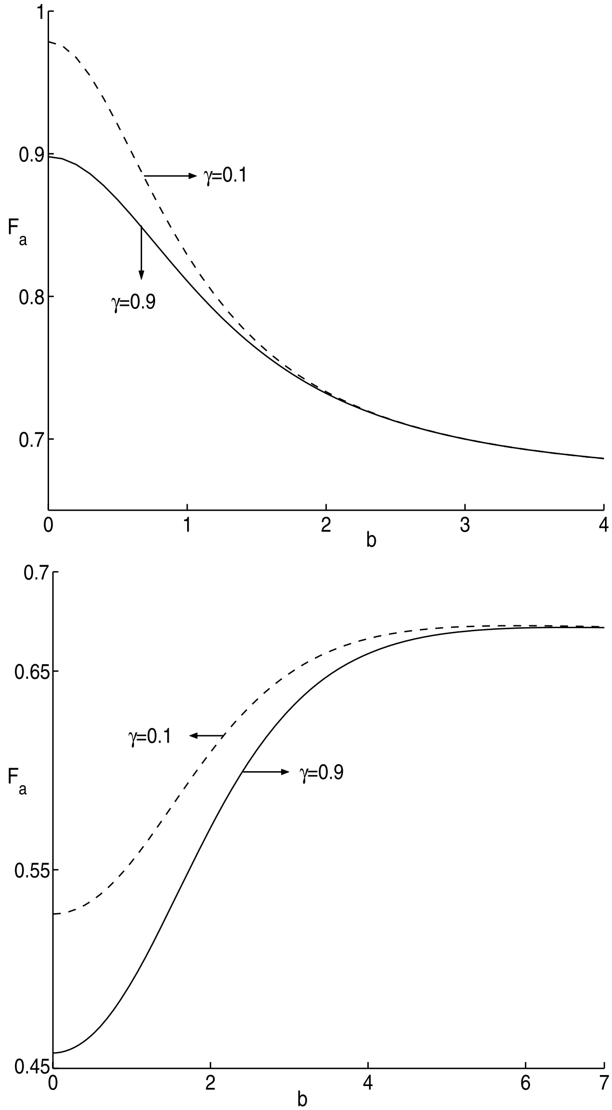


Fig. 2 The average fidelity F_a as a function of the inhomogeneous magnetic field b with different anisotropy parameter γ at different temperature T and the coupling constant J with $b = 0$ (the *top panel*: $T = 0.3$ and the *bottom panel*: $T = 1$). For all plots: we set $J = 1$, $B = 0.5$ and $J_z = 0.9$

In Fig. 4, the transfer state average fidelity F_a is plotted as a function of the external uniform magnetic field B under the different values of anisotropy parameter γ at a finite temperature $T = 0.5$, where we set $J = 1$, $J_z = 0.9$ and inhomogeneous magnetic field $b = 1$. It firstly confirm that increasing B the value of F_a is initially decreased and then increased to a constant value, and this constant value are always smaller than 2/3. So it means that the increasing of B can make F_a to be smaller than the maximum value of classical information. Another point we must mention is that decreasing the anisotropy parameter γ evidently not only can improve the average fidelity F_a but also can broaden the region where the transfer state average fidelity F_a is larger than the maximum value of classical information 2/3. So a lower value of γ can result in ideal average fidelity.

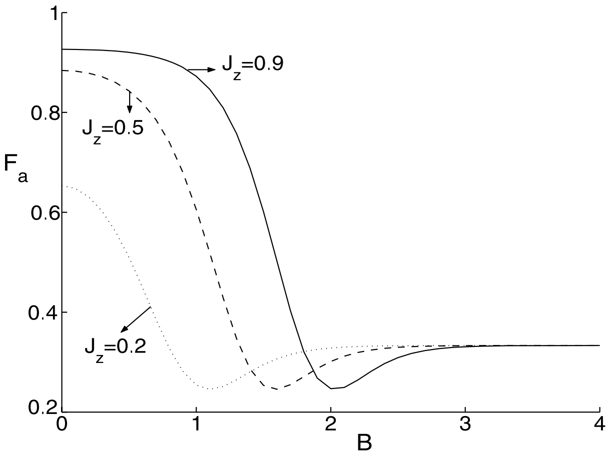


Fig. 3 With the coupling constant $J = 1$, for a finite value of nonuniform magnetic field $b = 0.5$. The average fidelity is plotted as a function of nonuniform magnetic field b with different values of J_z at a finite temperature $T = 0.2$. Where we set $\gamma = 0.9$

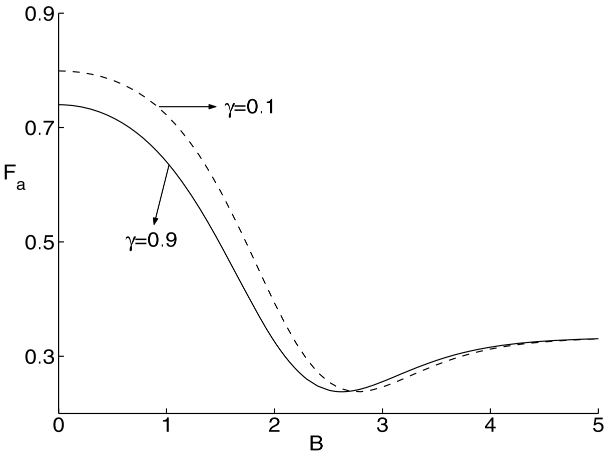


Fig. 4 With the coupling constant $J = 1$, the average fidelity is plotted as a function of the uniform magnetic field B with different values of γ at a finite temperature $T = 0.5$. Where we set $J_z = 0.9$ and $b = 1$

3 Conclusion

In conclusion, it is analyzed the transfer fidelities of two-qubit pure state via a two-qubit Heisenberg XYZ spin model under a nonuniform magnetic field. Our study shows that the average fidelity can be enhanced evidently by introducing the interaction of z component of two neighboring spins J_z. Increasing the value of J_z not only can improve the value of F_a, and result in ideal average fidelity (larger than the maximum value of classical communication 2/3) but also can broaden the region of ideal fidelity. Different the effect of J_z, the average fidelity can be improved only when the anisotropy parameter γ is decreased. Decreasing the value of γ also can broaden the region of idea F_a. So, a larger value of J_z or a smaller value of γ are all can result in an idea average fidelity. For a smaller value of J_z, F_a is increased with increasing the nonuniform field b, but for a greater value of J_z, F_a is decreased with increasing b. Increasing the uniform magnetic field B, the value of F_a having a tendency of initially decreasing and then improving. We also observe that for any finite temperature T, the value of F_a is infinitely close to the maximum value of classical communication 2/3 by increasing the inhomogeneous magnetic field b. This imply us that one can obtain the ideal fidelity by increasing b. However, for enough larger value of B, F_a

is always inferior to the classical communication. So, if one want an ideal transfer average fidelity, we can through several methods such as increasing J_z, decreasing anisotropy parameter γ, improving the inhomogeneous magnetic field b, or decreasing the external uniform magnetic field. F_a can be decreased by increasing the temperature T.

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